Integrals-tasks (III part) - **PARTIAL INTEGRATION (**Integration by parts)

If u and v are differentiable functions of x, then:

$$\boxed{\int udv = uv - \int vdu}$$

This method, partial integration, typically at the beginning of the study poorly understood. We will try, as far as

the written word it allows , to make you closer and explain method .

Given integral we compare with $\int udv$. "Something" (for example, Θ) select to be u , and " something" , for example Δdx, to be dv.

From what we choose to be u we are looking derivate , and from what we have chosen to be dv we are looking integral:

$$\left| \begin{array}{ll} \Theta = u & \Delta dx = dv \\ \Theta` dx = du & \int \Delta dx = v \end{array} \right|$$

When we find du and v ,we change that in formula of partial integration $uv - \int vdu$. The idea of partial integration is that $\int vdu$ must be easier than $\int udv$.

The most common **example** in which teachers explain the partial integration is:

Example 1. $\int xe^x dx = ?$

This integral compared with $\int udv$. Will choose that $x = u$ and e^x dx=dv.

$$\int xe^x dx = \left| \begin{array}{ll} x = u & e^x dx = dv \\ dx = du & \int e^x dx = v \\ & e^x = v \end{array} \right| = this \quad replace \quad in \quad u \cdot v - \int vdu$$

$$= \underbrace{x \cdot e^x}_{u \cdot v} - \underbrace{\int e^x dx}_{\int v \cdot du} = xe^x - e^x + C = \boxed{e^x (x-1) + C}$$

And what would happen if we choose wrong? Da vidimo: Let's see:

$$\int xe^x dx = \left| \begin{array}{ll} e^x = u & \mathrm{x}dx = dv \\ e^x dx = du & \int xdx = v \\ & \frac{x^2}{2} = v \end{array} \right| = \frac{x^2}{2} \cdot e^x - \boxed{\int \frac{x^2}{2} \cdot e^x dx} \rightarrow \text{This integral is "heavier" than the original!}$$

To be "smart" pick in the given integrals, we will share in the 4th group.

1. ***GROUP*** Here we choose the x or term associated with x equal to U, and everything else is dv

For example: $\int x\cos x dx$, $\int (1-x)\sin x dx$, $\int xe^{x}dx$, $\int \frac{x}{\sin^2 x}dx$, $\int (x^2-2x+5)e^{-x}dx$ ...

2. *GROUP* Here does not take x for u, but function in addition to x, (ie the term with x). lnx = u,

arcsinx = u, arctgx = u and the rest is dv.

For example: $\int x\ln x dx$ $\int x\arcsin x dx$, $\int x^2 arctgxdx$, $\int x^3 \ln x dx$ ...

3. ***GROUP*** Here we take dx=dv, and lnx = u or arcsinx = u or arctgx = u.....

For example: $\int \ln x dx$, $\int \ln^2 x dx$, $\int arctgxdx$, $\int \arcsin x dx$ ...

4. ***GROUP*** These circle integrals, who always have their " friend " through which it is integral to give back to the beginning ...

For example: : $\int e^x \sin x dx$, $\int e^x \cos x dx$, $\int \sin(\ln x)dx$, $\int \cos(\ln x)dx$ ...

From each group will do a few examples ...

Clearly, the example made $\int xe^x dx$ belongs to the first group.

$\boxed{\text{Example 2.}}$ $\int (1-x)\sin x dx = ?$

$$\int (1-x)\sin x dx = \left| \begin{array}{ll} 1-x=u & \text{sinxdx=dv} \\ -dx=du & \int \text{sinxdx} = v \\ & -\cos x = v \end{array} \right| = \underbrace{(1-x)(-\cos x)}_{u\cdot v} - \underbrace{\int (-\cos x)(-dx)}_{\int vdu} = \boxed{(x-1)\cos x - \sin x + C}$$

$\boxed{\text{Example 3.}}$ $\int \frac{xdx}{\cos^2 x} = ?$

$$\int \frac{xdx}{\cos^2 x} = \left| \begin{array}{ll} x=u & \frac{dx}{\cos^2 x} = dv \\ dx=du & \int \frac{dx}{\cos^2 x} = v \\ & tgx = v \end{array} \right| = x\cdot tgx - \int tgxdx =$$

$\int tgx dx = ?$

$$\int tgx dx = \int \frac{\sin x}{\cos x} dx = \left| \begin{array}{l} \cos x = t \\ -\sin x dx = dt \\ \sin x dx = -dt \end{array} \right| = \int \frac{-dt}{t} = -\ln|t| = -\ln|\cos x|$$

back to the task:

$$\int \frac{xdx}{\cos^2 x} = x \cdot tgx - \int tgx dx = x \cdot tgx - (-\ln|\cos x|) + C = \boxed{x \cdot tgx + \ln|\cos x| + C}$$

$\boxed{\text{Example 4.}}$ $\int x \ln x dx = ?$

Here is tempting to take that $x = u$, but it led us to a dead end …

This integral is from the second group:

$$\int x \ln x dx = \left| \begin{array}{ll} \ln x = u & \int x dx = v \\ \frac{1}{x} dx = du & \frac{x^2}{2} = v \end{array} \right| = \underbrace{\ln x \cdot \frac{x^2}{2}}_{u \cdot v} - \underbrace{\int \frac{x^2}{2} \cdot \frac{1}{x} dx}_{\int vdu} = \frac{x^2}{2} \cdot \ln x - \int \frac{x^{\cancel{2}}}{2} \cdot \frac{1}{\cancel{x}} dx =$$

$$= \frac{x^2}{2} \cdot \ln x - \frac{1}{2} \int x dx = \frac{x^2}{2} \cdot \ln x - \frac{1}{2} \cdot \frac{x^2}{2} + C = \boxed{\frac{x^2}{2} \cdot \ln x - \frac{x^2}{4} + C}$$

$\boxed{\text{Example 5.}}$ $\int x \cdot arctgx dx = ?$

$$\int x \cdot arctgx dx = \left| \begin{array}{ll} arctgx = u & \int x dx = v \\ \frac{1}{1+x^2} dx = du & \frac{x^2}{2} = v \end{array} \right| = arctgx \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2} dx = arctgx \cdot \frac{x^2}{2} - \frac{1}{2} \cdot \int \frac{x^2}{1+x^2} dx$$

$$\int \frac{x^2}{1+x^2} dx = ?$$

$$\int \frac{x^2}{x^2+1} dx = \int \frac{x^2+1-1}{x^2+1} dx = \int \frac{x^2+1}{x^2+1} dx - \int \frac{1}{x^2+1} dx = \int \frac{\cancel{x^2+1}}{\cancel{x^2+1}} dx - \int \frac{1}{x^2+1} dx$$

$$= \int dx - \int \frac{1}{x^2+1} dx = \boxed{x - arctgx}$$

Now, we have: $\int x \cdot arctgx dx = arctgx \cdot \frac{x^2}{2} - \frac{1}{2} \cdot \int \frac{x^2}{1+x^2} dx = \boxed{arctgx \cdot \frac{x^2}{2} - \frac{1}{2}(x - arctgx) + C}$

$\boxed{\text{Example 6.}}$ $\int \frac{x^3 \arccos x}{\sqrt{1-x^2}}dx = ?$

And this is the integral of the second group al is a bit heavier and has more work.

$$\int \frac{x^3 \arccos x}{\sqrt{1-x^2}}dx = \left| \begin{array}{ll} \arccos x = u & \frac{x^3}{\sqrt{1-x^2}}\mathrm{d}x = dv \\ -\frac{dx}{\sqrt{1-x^2}} = du & \boxed{\int \frac{x^3}{\sqrt{1-x^2}}\mathrm{d}x = v} \end{array} \right| = \text{ Framed integral will be solved "on side"}$$

$$\int \frac{x^3}{\sqrt{1-x^2}}\mathrm{d}x = \int \frac{\boxed{x^2} \cdot x}{\sqrt{1-x^2}}\mathrm{d}x = \left| \begin{array}{l} 1-x^2 = t^2 \\ -\cancel{2}xdx = \cancel{2}tdt \\ xdx = -tdt \\ 1-x^2 = t^2 \rightarrow x^2 = \boxed{1-t^2} \end{array} \right| = \int \frac{1-t^2}{\cancel{t}}(-\cancel{t}dt) = \int (t^2-1)dt = \frac{t^3}{3} - t = \frac{t^3-3t}{3} =$$

$$\frac{t(t^2-3)}{3} = \frac{\sqrt{1-x^2}(1-x^2-3)}{3} = \frac{\sqrt{1-x^2}(-x^2-2)}{3} = -\frac{\sqrt{1-x^2}(x^2+2)}{3}$$

Let's go back now in the partial integration:

$$\int \frac{x^3 \arccos x}{\sqrt{1-x^2}}dx = \left| \begin{array}{ll} \arccos x = u & \frac{x^3}{\sqrt{1-x^2}}\mathrm{d}x = dv \\ -\frac{dx}{\sqrt{1-x^2}} = du & -\frac{\sqrt{1-x^2}(x^2+2)}{3} = v \end{array} \right| =$$

$$= \arccos x \cdot \left(-\frac{\sqrt{1-x^2}(x^2+2)}{3}\right) - \int \left[-\frac{\cancel{\sqrt{1-x^2}}(x^2+2)}{3}\right]\left[-\frac{dx}{\cancel{\sqrt{1-x^2}}}\right]$$

$$= -\arccos x \cdot \left(\frac{\sqrt{1-x^2}(x^2+2)}{3}\right) - \frac{1}{3}\int (x^2+2)dx$$

$$= \boxed{-\arccos x \cdot \left(\frac{\sqrt{1-x^2}(x^2+2)}{3}\right) - \frac{1}{3}\left(\frac{x^3}{3} + 2x\right) + C}$$

$\boxed{\text{Example 7.}}$ $\int \ln x dx = ?$

This is integral in our group III.

$$\int \ln x dx = \left| \begin{array}{ll} \ln x = u & \text{dx=dv} \\ \frac{1}{x} dx = du & \int dx = v \\ & x = v \end{array} \right| = \ln x \cdot x - \int x \cdot \frac{1}{x} dx = x \ln x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} dx = x \ln x - x + C = \boxed{x(\ln x - 1) + C}$$

$\boxed{\text{Example 8.}}$ $\int \ln^2 x dx = ?$

$$\int \ln^2 x dx = \left| \begin{array}{ll} \ln^2 x = u & \text{dx=dv} \\ ?\, dx = du & \int dx = v \\ & x = v \end{array} \right| , \quad ? = ?$$

as is $(\ln^2 x)` = 2 \ln x \cdot (\ln x)` = 2 \ln x \cdot \frac{1}{x} = \frac{2 \ln x}{x}$

Let's go back to the task:

$$\int \ln^2 x dx = \left| \begin{array}{ll} \ln^2 x = u & \text{dx=dv} \\ \frac{2 \ln x}{x} dx = du & \int dx = v \\ & x = v \end{array} \right| = \ln^2 x \cdot x - \int x \cdot \frac{2 \ln x}{x} dx = x \cdot \ln^2 x - 2 \int \cancel{x} \cdot \frac{\ln x}{\cancel{x}} dx = x \cdot \ln^2 x - 2 \boxed{\int \ln x dx}$$

We worked and got $\int \ln x dx$, That we have decided in the previous example. So here we would have to do a new partial integration!

We will use the solution to the previous example $\int \ln x dx = x(\ln x - 1)$

So the solution of our integral is:

$$\int \ln^2 x dx = x \cdot \ln^2 x - 2 \boxed{\int \ln x dx} = x \cdot \ln^2 x - 2x(\ln x - 1) + C = \boxed{x \cdot (\ln^2 x - 2 \ln x + 2) + C}$$

$\boxed{\text{Example 9.}}$ $\int \ln(x+\sqrt{1+x^2})dx = ?$

This is a difficult task and we will have more work ...

$$\int \ln(x+\sqrt{1+x^2})dx = \left| \begin{matrix} \ln(x+\sqrt{1+x^2}) = u & dx = dv \\ ?\, du & x = v \end{matrix} \right| ,$$?=? To find the derivative:

$$[\ln(x+\sqrt{1+x^2})]` = \frac{1}{x+\sqrt{1+x^2}} \cdot (x+\sqrt{1+x^2})` = \frac{1}{x+\sqrt{1+x^2}} \cdot (1+\frac{1}{2\sqrt{1+x^2}} \cdot (1+x^2)`)$$

$$= \frac{1}{x+\sqrt{1+x^2}} \cdot (1+\frac{1}{\cancel{2}\sqrt{1+x^2}} \cdot \cancel{2}x)$$

$$= \frac{1}{x+\sqrt{1+x^2}} \cdot (1+\frac{x}{\sqrt{1+x^2}})$$

$$= \frac{1}{\cancel{x+\sqrt{1+x^2}}} \cdot (\frac{\cancel{\sqrt{1+x^2}+x}}{\sqrt{1+x^2}}) = \boxed{\frac{1}{\sqrt{1+x^2}}}$$

Let's go back to the task:

$$\int \ln(x+\sqrt{1+x^2})dx = \left| \begin{matrix} \ln(x+\sqrt{1+x^2}) = u & dx = dv \\ \frac{1}{\sqrt{1+x^2}} dx = du & x = v \end{matrix} \right| = \ln(x+\sqrt{1+x^2}) \cdot x - \int x \cdot \frac{1}{\sqrt{1+x^2}} dx =$$

$$= \ln(x+\sqrt{1+x^2}) \cdot x - \boxed{\int \frac{x}{\sqrt{1+x^2}} dx} =$$

Again the problem, draw a framed integral and solve it using substitution:

$$\int \frac{x}{\sqrt{1+x^2}} dx = \left| \begin{matrix} \sqrt{1+x^2} = t \\ \frac{x}{\sqrt{1+x^2}} dx = dt \end{matrix} \right| = \int dt = t = \sqrt{1+x^2}$$

Finally, the solution will be:

$$\int \ln(x+\sqrt{1+x^2})dx = \boxed{x \cdot \ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C}$$

And to show a few examples from the IV group:

$\boxed{\text{Example 10.}}$ $\int \sin(\ln x)dx = ?$

Start with partial integration (initial integral is usually marked with *I):*

$$I = \int \sin(\ln x)dx = \left| \begin{matrix} \sin(\ln x) = u & dx = dv \\ \cos(\ln x)\cdot(\ln x)`dx = du & x = v \\ \cos(\ln x)\cdot\frac{1}{x}dx & \end{matrix} \right| =$$

$$= \sin(\ln x)\cdot x - \int \cancel{x}\cdot\cos(\ln x)\cdot\frac{1}{\cancel{x}}dx = \sin(\ln x)\cdot x - \int \cos(\ln x)dx$$

For now : $\boxed{I = \sin(\ln x)\cdot x - \int \cos(\ln x)dx}$

*

Integral $\int \cos(\ln x)dx$ again solve with partial integration:

$$\int \cos(\ln x)dx = \left| \begin{matrix} \cos(\ln x) = u & \text{dx=dv} \\ \text{-sin(lnx)}\frac{1}{x}dx = du & x = v \end{matrix} \right| =$$

$$\cos(\ln x)\cdot x - \int \cancel{x}(\text{-sin(lnx)}\frac{1}{\cancel{x}})dx = \cos(\ln x)\cdot x + \int \sin(\ln x)dx = \cos(\ln x)\cdot x + I$$

So, we have : $\boxed{\int \cos(\ln x)dx = \cos(\ln x)\cdot x + I}$

Let's go back to the *

$I = \sin(\ln x)\cdot x - \int \cos(\ln x)dx$, here we replace $\int \cos(\ln x)dx = \cos(\ln x)\cdot x + I$

$$I = \sin(\ln x)\cdot x - [\cos(\ln x)\cdot x + I]$$

$$I = \sin(\ln x)\cdot x - \cos(\ln x)\cdot x - I$$

$$I + I = \sin(\ln x)\cdot x - \cos(\ln x)\cdot x$$

$$2I = x\cdot[\sin(\ln x) - \cos(\ln x)]$$

$$\boxed{I = \frac{x\cdot[\sin(\ln x) - \cos(\ln x)]}{2} + C}$$

Add a constant C only when *I* express

Most professors like to explain this type of integrals in the integrals:

$\int e^x \sin x dx$ and $\int e^x \cos x dx$

We will do a more general example:

Example 11. $\int e^{ax} \sin bx dx = ?$

$$I = \int e^{ax} \sin bx dx = \left| \begin{array}{ll} \sin bx = u & e^{ax} dx = dv \\ \cos bx \cdot (bx)` dx = du & \int e^{ax} dx = v \\ b \cos bx dx = du & \frac{1}{a} e^{ax} = v \end{array} \right| =$$

$$= \sin bx \cdot \frac{1}{a} e^{ax} - \int \frac{1}{a} e^{ax} b \cos bx dx = \frac{e^{ax} \sin bx}{a} - \frac{b}{a} \int e^{ax} \cos bx dx$$

For now, we have: $I = \frac{e^{ax} \sin bx}{a} - \frac{b}{a} \boxed{\int e^{ax} \cos bx dx}$

Solve $\int e^{ax} \cos bx dx$, and we will return that to the solution…

$$\int e^{ax} \cos bx dx = \left| \begin{array}{ll} \cos bx = u & e^{ax} dx = dv \\ -\sin bx \cdot (bx)` dx = du & \int e^{ax} dx = v \\ -b \sin bx dx = du & \frac{1}{a} e^{ax} = v \end{array} \right| =$$

$$= \cos bx \cdot \frac{1}{a} e^{ax} - \int \frac{1}{a} e^{ax} (-b \sin bx) dx = \frac{e^{ax} \cos bx}{a} + \frac{b}{a} \int e^{ax} \sin bx dx$$

$So: \int e^{ax} \cos bx dx = \frac{e^{ax} \cos bx}{a} + \frac{b}{a} \int e^{ax} \sin bx dx$ or

$$\boxed{\int e^{ax} \cos bx dx = \frac{e^{ax} \cos bx}{a} + \frac{b}{a} \cdot I}$$

$$I = \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax}\cos bx dx$$

$$I = \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\cdot\left(\frac{e^{ax}\cos bx}{a} + \frac{b}{a}\cdot I\right) \quad \text{here we have to express } I$$

$$I = \frac{e^{ax}\sin bx}{a} - \frac{b\cdot e^{ax}\cos bx}{a^2} - \frac{b^2}{a^2}\cdot I \quad ..../\bullet a^2$$

$$a^2\cdot I = a\cdot e^{ax}\sin bx - b\cdot e^{ax}\cos bx - b^2\cdot I$$

$$a^2\cdot I + b^2\cdot I = a\cdot e^{ax}\sin bx - b\cdot e^{ax}\cos bx$$

$$I(a^2+b^2) = a\cdot e^{ax}\sin bx - b\cdot e^{ax}\cos bx$$

$$I = \frac{a\cdot e^{ax}\sin bx - b\cdot e^{ax}\cos bx}{a^2+b^2}$$

$$\boxed{I = \frac{e^{ax}(a\cdot\sin bx - b\cdot\cos bx)}{a^2+b^2} + C}$$

The solution to this generalized integrals can be applied to solve such $\int e^x \sin x dx$. *How?*

For $a=1$ and $b=1$ is $\frac{e^{ax}(a\cdot\sin bx - b\cdot\cos bx)}{a^2+b^2} = \frac{e^{1x}(1\cdot\sin 1x - 1\cdot\cos 1x)}{1^2+1^2} = \boxed{\frac{e^x(\sin x - \cos x)}{2}}$

So : $\int e^x \sin x dx = \frac{e^x(\sin x - \cos x)}{2} + C$

Example 12. $\int\sqrt{a^2-x^2}\,dx = ?$

This is one of the most famous integrals which can be solved in several ways.

Let's see how this would work out with the partial integration …

First, we must do a "little" rationalization:

$$\sqrt{a^2-x^2} = \frac{\sqrt{a^2-x^2}}{1}\cdot\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} = \frac{a^2-x^2}{\sqrt{a^2-x^2}} = \frac{a^2}{\sqrt{a^2-x^2}} - \frac{x^2}{\sqrt{a^2-x^2}}$$

So, now we have two integrals (the start integral we will mark with *I*)

$$I = \int\sqrt{a^2-x^2}\,dx = \int\frac{a^2}{\sqrt{a^2-x^2}}dx - \int\frac{x^2}{\sqrt{a^2-x^2}}dx$$

The first of these is tablet: $\int \frac{a^2}{\sqrt{a^2-x^2}}dx = a^2\int \frac{dx}{\sqrt{a^2-x^2}} = a^2 \cdot \arcsin\frac{x}{a}$

A second will solve with partial integration:

$$\int \frac{x^2}{\sqrt{a^2-x^2}}dx = \left| \begin{array}{ll} x = u & \frac{x}{\sqrt{a^2-x^2}}dx = dv \\ dx = du & \int \frac{x}{\sqrt{a^2-x^2}}dx = v \end{array} \right| =$$

First to solve:

$$\int \frac{x}{\sqrt{a^2-x^2}}dx = \left| \begin{array}{l} a^2-x^2 = t^2 \\ -\cancel{2}xdx = \cancel{2}tdt \\ xdx = -tdt \end{array} \right| = \int \frac{-\cancel{t}dt}{\cancel{t}} = -t = -\sqrt{a^2-x^2}$$

Now go back:

$$\int \frac{x^2}{\sqrt{a^2-x^2}}dx = \left| \begin{array}{ll} x = u & \frac{x}{\sqrt{a^2-x^2}}dx = dv \\ dx = du & \int \frac{x}{\sqrt{a^2-x^2}}dx = v \\ & -\sqrt{a^2-x^2} = v \end{array} \right| = -x\sqrt{a^2-x^2} - \int(-\sqrt{a^2-x^2})dx$$

$$\int \frac{x^2}{\sqrt{a^2-x^2}}dx = -x\sqrt{a^2-x^2} + \int(\sqrt{a^2-x^2})dx$$

$$\int \frac{x^2}{\sqrt{a^2-x^2}}dx = -x\sqrt{a^2-x^2} + I$$

To remember the beginning:

$$I = \int \frac{a^2}{\sqrt{a^2-x^2}}dx - \int \frac{x^2}{\sqrt{a^2-x^2}}dx$$

$$I = a^2 \cdot \arcsin\frac{x}{a} - (-x\sqrt{a^2-x^2} + I)$$

$$I = a^2 \cdot \arcsin\frac{x}{a} + x\sqrt{a^2-x^2} - I$$

$I + I = a^2 \cdot \arcsin\frac{x}{a} + x\sqrt{a^2-x^2} \rightarrow 2I = a^2 \cdot \arcsin\frac{x}{a} + x\sqrt{a^2-x^2}$ and finally:

$$\boxed{I = \frac{1}{2}\left(a^2 \cdot \arcsin\frac{x}{a} + x\sqrt{a^2-x^2}\right) + C}$$